

# On Fair Compromise in Price Matching Engines

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## Abstract

We show a fair compromise between two prices is the geometric mean of the two, not the arithmetic mean.

Two parties,  $A$  and  $B$ , wish to trade.  $A$  has  $u_1$ s and  $B$  has  $u_2$ s. They determine that an agreeable price should have one important property: that they are benefited equally by the trade. They further decide the benefit should be measured in capital growth. Because they wish to derive their price without external bias, they accept a fair price must be judged against their own expectations. In order to determine some measure of growth, they agree it can be replaced with their improvement over their worst acceptable situation, provided it was actually an improvement<sup>1</sup>. This represents the perceived benefit or incentive for each party.

$$\frac{A_{actual,end}}{A_{expected,end}} = \frac{B_{actual,end}}{B_{expected,end}} \quad (1)$$

They each have a worst-case price in their head, such that  $p_a > p_b$  in units of  $u_1/u_2$  (where  $A$  is buying  $u_2$ s, and  $B$  is selling them, so  $A$  is willing to pay more than  $B$  is asking). This inequality ensures that  $A$  always gets *more*  $u_2$ s than they expect, and  $B$  always gives away less than expected (per  $u_1$ , that is). Thus, both parties gain.

They agree there exists some fair price  $\bar{p}$  such that  $p_a > \bar{p} > p_b$ .

They also agree the following relationship between start and end balances holds true:

$$\frac{A_{u_1,start}}{\bar{p}} = A_{u_2,end}; \quad B_{u_2,start}\bar{p} = B_{u_1,end} \quad (2)$$

Since  $p_{actual} = \bar{p}$ , and  $p_{expected} = p_a$  (or  $p_b$ ), (1) and (2) give us

$$\frac{\frac{A_{u_1,start}}{\bar{p}}}{\frac{A_{u_1,start}}{p_a}} = \frac{B_{u_2,start}\bar{p}}{B_{u_2,start}p_b} \quad (3)$$

$$\bar{p} = \sqrt{p_a p_b} \quad (4)$$

Thus, we see with (4) that a fair price is the geometric mean of both prices.

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<sup>1</sup>This isn't technically required, if a trade needed to occur this method also provides a valid conclusion, however, this is being considered in the context of a matching engine.

### Arithmetic Mean

If we use the arithmetic mean instead of the geometric mean, we set  $\bar{p} = \frac{p_a + p_b}{2}$ , and thus using (1) and (2) we see that instead of (3) we find:

$$\begin{aligned} LHS &= \frac{p_a}{\frac{p_a + p_b}{2}} \\ &= \frac{2p_a}{p_a + p_b} \\ &= \frac{4p_a p_b}{2p_b(p_a + p_b)} \\ RHS &= \frac{p_a + p_b}{2p_b} \\ &= \frac{(p_a + p_b)^2}{2p_b(p_a + p_b)} \\ &\text{but} \\ 2p_b(p_a + p_b)(RHS - LHS) &= (p_a + p_b)^2 - 4p_a p_b \\ &= (p_a - p_b)^2 \\ &> 0 \text{ if } p_a \neq p_b \\ \text{Therefore,} \\ 2p_b(p_a + p_b)(RHS - LHS) &> 0 \\ RHS - LHS &> 0 \\ RHS &> LHS \end{aligned} \tag{5}$$

This inequality shows that the person with the *lower* price is disproportionately benefited (gains more) by using the arithmetic mean.

Additionally, the arithmetic average of  $p_a$  and  $p_b$  is not equal to the reciprocal of the average of  $\frac{1}{p_a}$  and  $\frac{1}{p_b}$ ; their geometric averages, however, have this property. In a matching engine this means there is no bias provided by the base currency pair.

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